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An exact method on penny-shaped cracked homogeneous and composite cylinders

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Abstract

For three-dimensional mode I crack problems, a new concept on the crack surface widening energy release rate and its definition in form of integral are proposed, which unlike the classical crack extension energy release rate. This energy release rate is related to the stress intensity factor by application of the principle of virtual work. From the present discussions, a very simple method is established to estimate stress intensity factors for three-dimensional cracks within the framework of elementary strength theory of materials. A series of exact solutions are derived for penny-shaped cracked homogeneous and composite cylinders. Some results are shown to agree well with those available in the existing literature. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Stress intensity factor; Fracture; Mechanics

1. Introduction

The J integral has been used to analyze crack problems for long time. Actually, if crack surface is parallel to axis x_1, J integral is the first component J_1 of two-dimensional conservation law J_k $(k = 1, 2)$ (Eshelby, 1951; Sih, 1969; Budiansky and Rice, 1973; Rice, 1968). During the last several decades, one focused attention nearly only on the applications of J or J_1 until the approaches of Kienzler and Herrmann (1986a,b). In the recent years, G^* or J_2 integral, the second component of conservation law J_k, and its applications were proposed (Xie et al., 1998a,b; Xie, 1998, 2000). J and G^* integrals have different physical meanings when applied to solve crack problems. J integral means the crack extension energy release rate and G^* integral the crack mouth widening energy release rate for the same crack problem. The most important is that the two integrals yield the $K_{\rm I}$ using different components of displacements and stresses. Take the cracked beams for example. The strain and stress fields needed in J integral method should be usually

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determined by numerical analysis; but the strain and stress fields needed in G^* integral method can be given only by elementary mechanics or bending theory (Xie et al., 1998a).

In what follows, an exact method is proposed and a series of exact solutions are derived for penny-shaped cracked homogeneous and composite cylinders using the G^* integral.

2. G^* integral

Based on three-dimensional conservation law (Eshelby, 1951; Sih, 1969; Budiansky and Rice, 1973), the three-dimensional crack surface widening energy release rate can be defined by the integral

$$G^* = \iint_A (wn_2 - T_i u_{i,2})\,\mathrm{d}A \tag{1}$$

Note that A is one of the two crack surfaces, the upper surface whose axis x_2 is perpendicular to, is shown in Fig. 1. The integration area A can also be a curved surface A' in Fig. 1(b). Boundary of A' is the rim of the crack. Surfaces A and A' form a closed curved. In Eq. (1) w denotes the energy density; T_i the traction vector; u_i the displacements and n_i the outward unit normal. The physical meaning of Eq. (1) is the energy release rate due to the upper crack surface A moving in x_2 direction with unit translation as the lower remains stationary similar to literature (Xie et al., 1998a).

For a two-dimensional deformation field, the counterpart of Eq. (1), the crack mouth widening energy release rate (Xie et al., 1998a), is

$$G^* = \int_s (wn_2 - T_i u_{i,2})\,\mathrm{d}s \tag{2}$$

where s in Eq. (2) is a curve in the x_1, x_2 plane. For mode I crack with a unit thickness, shown in Fig. 2, let s in Eq. (2) represent to the path s_{efg} next to the upper crack tip region. Note that s_{ef} is a straight line and s_{fg} is a quarter of a circle. Along the path s_{ef} and s_{fg}, Eq. (2) yields (Xie et al., 1998a)

$$\int_{s_{\mathrm{ef}}} (wn_2 - T_i u_{i,2})\,\mathrm{d}s = 0 \tag{3}$$

and

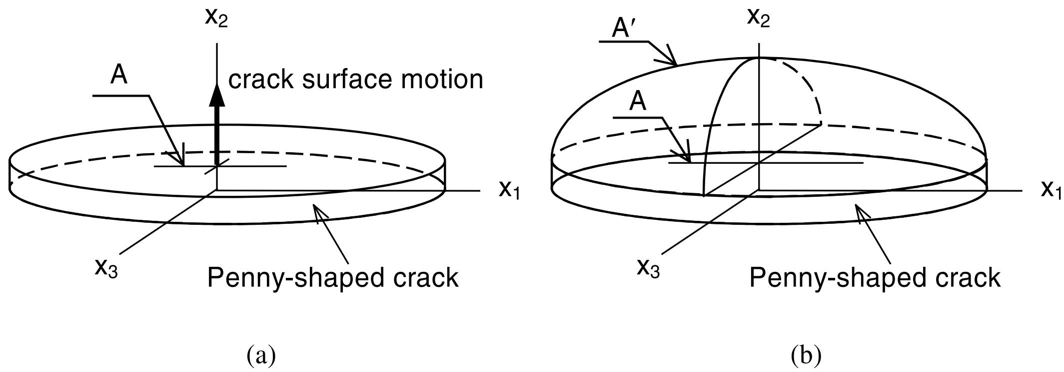

Fig. 1. Upper crack surface moves in x_2 direction.

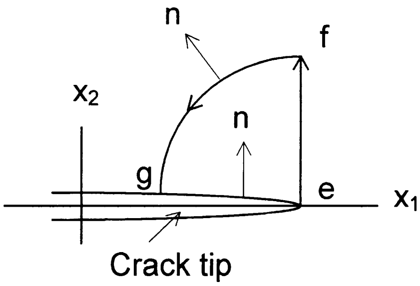

Fig. 2. Integration path for two-dimensional Mode I crack.

$$\int_{s_{\rm fg}} (wn_2 - T_i u_{i,2})\,{\rm d}s = \frac{(1-v^2)K_{\rm I}^2}{2\pi E} \quad \text{(plane strain)} \tag{4}$$

For a closed path $s = s_{\rm ef} + s_{\rm fg} - s_{\rm eg}$, $\oint_s (wn_2 - T_i u_{i,2})\,{\rm d}s = 0$. It follows that

$$\int_{s_{\rm eg}} (wn_2 - T_i u_{i,2})\,{\rm d}s = \int_{s_{\rm efg}} (wn_2 - T_i u_{i,2})\,{\rm d}s = \frac{(1-v^2)K_{\rm I}^2}{2\pi E} \tag{5}$$

Eq. (5) can be interpreted as energy release per unit moving of boundary $s_{\rm eg}$ in x_2 direction or the crack mouth widening energy release rate. v is Poisson's ratio and E the elastic modulus.

3. Penny-shaped cracked cylinders under tension

3.1. Energy release rate expressed by G^* integral

A cylinder with a penny-shaped crack of radius a subjected to tension is shown in Fig. 3. Along the front of crack tip, there is a stress field of plane strain. Refer to Fig. 4 for considering the integral surface. Let A in Eq. (1) be the upper crack surface and divide A into two regions. One region denoted by A_1 is $x_1^2 + x_3^2 < (a-r)^2$, $x_2 = 0^+$. The other denoted by A_2 is $(a-r)^2 \leqslant x_1^2 + x_3^2 \leqslant a^2$, $x_2 = 0^+$. The surface formed by revolving path $s_{\rm efg}$ around axis x_2 is expressed by A_2'. A_2' and A_2 can form a closed surface. Then the crack surface widening energy release rate for penny-shaped crack can be expressed by

$$G^* = \iint_A (wn_2 - T_i u_{i,2})\,{\rm d}A = \iint_{A_1} w\,{\rm d}A + \iint_{A_2} (wn_2 - T_i u_{i,2})\,{\rm d}A \tag{6}$$

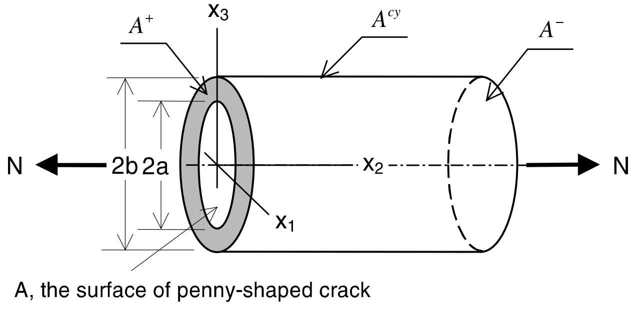

Fig. 3. Cracked cylinder with one-half symmetry subjected to tension (Mode I).

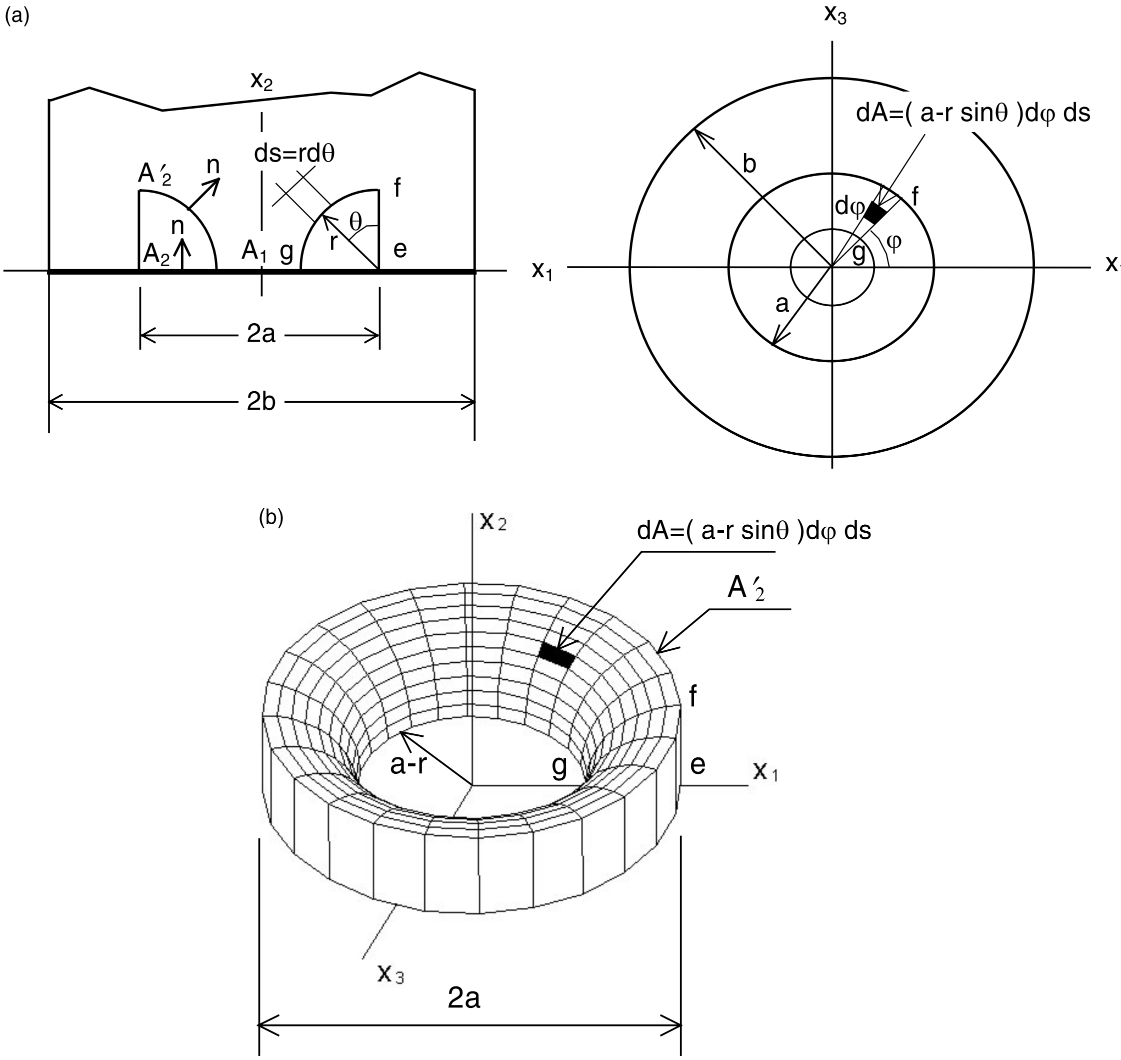

Fig. 4. Cracked cross-sectional area and integration surface. (a) Integration curved surfaces; (b) local integration surface A'_2 formed by revolving path s_{efg} around axis x_2.

From the conservation law, i.e., $\oint_{A'_2-A_2}(wn_2 - T_i u_{i,2})\,\mathrm{d}A = 0$ for the closed surface $A'_2 - A_2$ and Eqs. (4) and (5), it follows that

$$\iint_{A_2}(wn_2 - T_i u_{i,2})\,\mathrm{d}A = \iint_{A'_2}(wn_2 - T_i u_{i,2})\,\mathrm{d}A = 2\pi\int_{s_{\mathrm{fg}}}(wn_2 - T_i u_{i,2})(a - r\sin\theta)\,\mathrm{d}s \overset{r/a\to 0}{=} \frac{a(1-\nu^2)K_{\mathrm{I}}^2}{E} \tag{7}$$

Then Eq. (6) becomes

$$G^* = \iint_A (wn_2 - T_i u_{i,2})\,\mathrm{d}A = \iint_{A_1} w\,\mathrm{d}A + \frac{a(1-\nu^2)K_{\mathrm{I}}^2}{E} \tag{8}$$

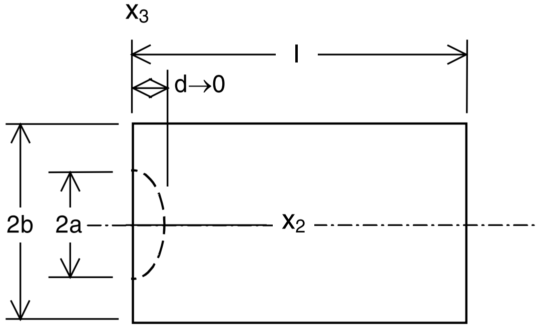

Fig. 5. Oblate spheroid model for penny-shaped crack.

3.2. Energy release rate based on elementary mechanics

When $d \to 0$ in Fig. 5, the oblate spheroid will become a penny-shaped crack in cylinder. The strain energy of the cracked cylinder can be expressed by

$$\begin{aligned} U &= \int_0^{l-d} \frac{N^2}{2ES}\,\mathrm{d}\bar{x}_2 + \int_0^d \frac{N^2\,\mathrm{d}x_2}{2ES[1-(a/b)^2(1-x_2^2/d^2)]} \\ &= \frac{N^2}{2ES}(l-d) + \frac{N^2 d}{2ES}\frac{1}{(a/b)\sqrt{1-(a/b)^2}}\arctan\frac{a/b}{\sqrt{1-(a/b)^2}}, \quad S=\pi b^2 \end{aligned} \tag{9}$$

where the origin of $\bar{x}_2$ coordinate is at the right end of cylinder.

Making use of Clapeyron's theorem and the external work, $V = 2U$ and the total potential energy $\Pi = U - V = -U$. From elementary theory, the crack surface widening energy release rate is

$$G^* = \lim_{d\to 0}\left(-\frac{\partial \Pi}{\partial d}\right) = \frac{N^2}{2ES}\left[\frac{1}{(a/b)\sqrt{1-(a/b)^2}}\arctan\frac{a/b}{\sqrt{1-(a/b)^2}} - 1\right] \tag{10}$$

3.3. Stress intensity factor

Eqs. (8) and (10) are the crack surface widening energy release rate calculated by different definitions, but the results should be the same. Let Eq. (8) be equal to Eq. (10), it follows that

$$\frac{a(1-\nu^2)K_\mathrm{I}^2}{E} + \iint_{A_1} w\,\mathrm{d}A = \frac{N^2}{2ES}\left[\frac{1}{(a/b)\sqrt{1-(a/b)^2}}\arctan\frac{a/b}{\sqrt{1-(a/b)^2}} - 1\right] \tag{11}$$

Because of free action of crack surface, the integral in left-hand side of Eq. (11) is a small quantity, which can be neglected. Hence, the stress intensity factor is

$$K_\mathrm{I} = \sigma_n\sqrt{\pi(b-a)}F\left(\frac{a}{b}\right) \tag{12}$$

where the normalized stress intensity factor is

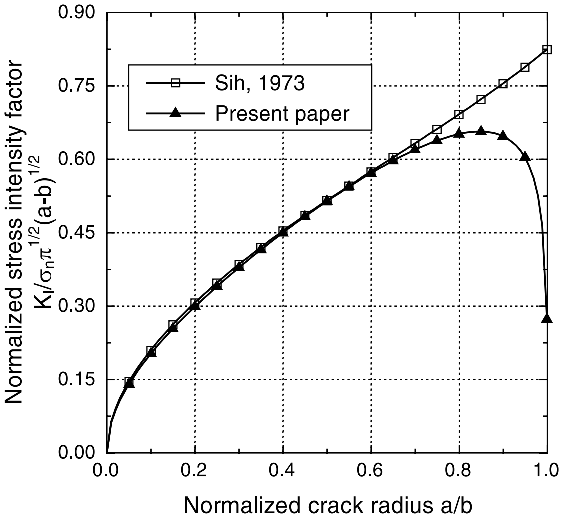

Fig. 6. Variations of normalized stress intensity factor with normalized crack radius for cylinder with penny-shaped crack under tension.

$$F\left(\frac{a}{b}\right) = \sqrt{\frac{1-a/b}{a/b}}\left(1+\frac{a}{b}\right)\left\{\frac{1}{2(1-\nu^2)}\left[\frac{1}{(a/b)\sqrt{1-(a/b)^2}}\arctan\frac{a/b}{\sqrt{1-(a/b)^2}} - 1\right]\right\}^{1/2} \tag{13}$$

and

$$\sigma_n = \frac{N}{\pi(b^2-a^2)} \tag{14}$$

Fig. 6 compares the results of Eq. (13) with those in literature (Sih, 1973).

4. Penny-shaped cracked cylinders under lateral loads

4.1. Energy release rate expressed by G^* integral

Referring to Fig. 7, the center-cracked cylinder deflects in x_2, x_3 plane and the lateral loads are added in the same plane. q is the transverse load per unit length; Q the transverse shear force ($Q^+ = 0$ for Mode I crack); M the bending moment. The symbol '+' denotes the cracked cross-section; '−' the remote uncracked cross-section. At the bottom point of crack, the stress intensity factor reaches its maximum value $K_{\mathrm{I\,max}}$. Along the crack tip front, the stress intensity factor K_I varies with angular position φ according to $K_\mathrm{I} = -K_{\mathrm{I\,max}}\sin\varphi$ (see Fig. 4). Similar to Eqs. (6)–(8), the crack surface widening energy release rate derived from the concept of G^* integral can be expressed as

$$G^* = \iint_A (wn_2 - T_i u_{i,2})\,\mathrm{d}A = \iint_{A_1} w\,\mathrm{d}A + \frac{a(1-\nu^2)K_{\mathrm{I\,max}}^2}{2E} \tag{15}$$

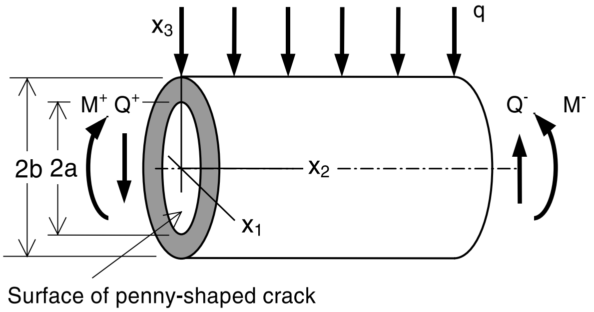

Fig. 7. Cracked cylinder with one-half symmetry subjected to lateral loads (Mode I).

4.2. Energy release rate based on elementary mechanics

Refer to Figs. 5 and 7 to calculate the strain energy of the cracked cylinder. It is not difficult to get.

$$U = \int_0^{l-d} \frac{(M^- + Q^-\bar{x}_2 - 0.5q\bar{x}_2^2)^2}{2EI}\,\mathrm{d}\bar{x}_2 + \int_0^d \frac{\left(M^+ - Q^+x_2 - 0.5qx_2^2\right)^2}{2EI\bar{\gamma}(a/b, x_2/d)}\,\mathrm{d}x_2 \tag{16}$$

where

$$\bar{\gamma}\left(\frac{a}{b}, \frac{x_2}{d}\right) = 1 - \left(\frac{a}{b}\right)^4 \left(1 - \frac{x_2^2}{d^2}\right)^2 \tag{17}$$

Then the crack surface widening energy release rate under the condition of elementary theory is

$$G^* = \lim_{d\to 0}\left(-\frac{\partial \Pi}{\partial d}\right) = \frac{(M^+)^2}{2EI}\gamma\left(\frac{a}{b}\right) - \frac{(M^- + Q^- l - 0.5ql^2)^2}{2EI} \tag{18}$$

where

$$\gamma\left(\frac{a}{b}\right) = \int_0^1 \frac{\mathrm{d}t}{1 - (a/b)^4(1 - t^2)^2}$$

$$= \frac{1}{4}\frac{1}{a/b\sqrt{1 + (a/b)^2}} \ln \frac{\sqrt{1 + (a/b)^2} + a/b}{\sqrt{1 + (a/b)^2} - a/b} + \frac{1}{2}\frac{1}{a/b\sqrt{1 - (a/b)^2}} \arctan \frac{a/b}{\sqrt{1 - (a/b)^2}} \tag{19}$$

From equilibrium conditions, Eq. (18) can be rearranged as

$$G^* = \lim_{d\to 0}\left(-\frac{\partial \Pi}{\partial d}\right) = \frac{(M^+)^2}{2EI}\left[\gamma\left(\frac{a}{b}\right) - 1\right] \tag{20}$$

which shows that the crack surface widening energy release rate depends only on the bending moment in cracked cross-section of the cylinders.

4.3. Stress intensity factor

Let Eq. (15) be equal to Eq. (20). It gives

$$K_{\mathrm{I}\max} = \sigma_n\sqrt{\pi(b - a)}F\left(\frac{a}{b}\right) \tag{21}$$

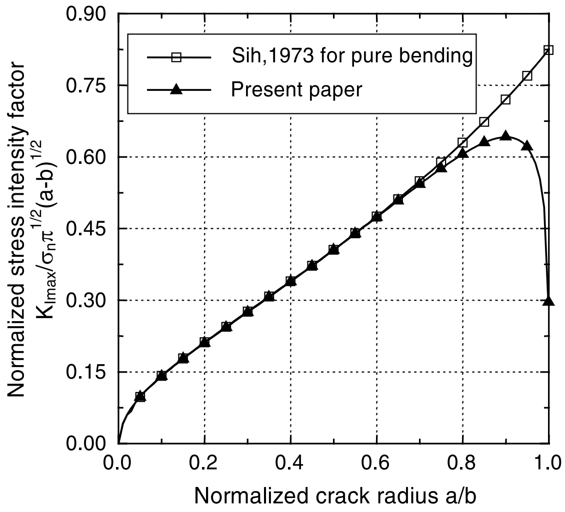

Fig. 8. Variations of normalized stress intensity factor with normalized crack radius for cylinder with penny-shaped crack under bending.

where the normalized stress intensity factor

$$F\left(\frac{a}{b}\right) = \frac{1-(a/b)^4}{(a/b)^{3/2}(1-a/b)^{1/2}}\left\{\frac{1}{1-\nu^2}\left[\gamma\left(\frac{a}{b}\right)-1\right]\right\}^{1/2} \tag{22}$$

and

$$\sigma_n = \frac{4M^+a}{\pi(b^4-a^4)} \tag{23}$$

Fig. 8 compares the results of present paper with one from literature (Sih, 1973) for the case of pure bending.

The most interested is that, for pure bending, distributed loads, three-point bending or combined loads, one may easily get the stress intensity factor for center cracked cylinders, as long as knowing the bending moment in the cracked cross-section.

5. Penny-shaped cracked composite cylinders

A cracked composite cylinder subjected to tension is shown in Fig. 9. If $a < b - \Delta$, the G^* integral along crack surface A can be given by

$$G^* = \iint_A (wn_2 - T_i u_{i,2})\,\mathrm{d}A = \iint_{A_1} w\,\mathrm{d}A + \frac{a\left(1-\nu_2^2\right)K_\mathrm{I}^2}{E_2} \tag{24}$$

Refer to Figs. 5 and 9. The strain energy of the cracked composite cylinder can be expressed by

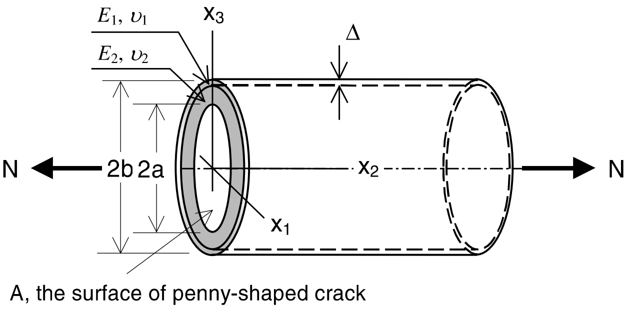

Fig. 9. Cracked composite cylinder with one-half symmetry subjected to tension (Mode I).

$$\begin{aligned} U &= \int_0^{l-d} \frac{N^2\,\mathrm{d}\bar{x}_2}{2(E_1S_1+E_2S_2)} + \int_0^{d} \frac{N^2\,\mathrm{d}x_2}{2(E_1S_1+E_2S_2)\hat{\gamma}(a/b,x_2/d)} \\ &= \frac{N^2(l-d)}{2(E_1S_1+E_2S_2)} + \frac{N^2 d}{2(E_1S_1+E_2S_2)}\int_0^1 \frac{\mathrm{d}t}{\hat{\gamma}(a/b,t)} \end{aligned} \tag{25}$$

where

$$\hat{\gamma}\left(\frac{a}{b},t\right) = 1 - \frac{1}{\lambda^2}\left(\frac{a}{b}\right)^2(1-t^2),$$

and

$$\lambda = \left(1+\frac{E_1S_1}{E_2S_2}\right)^{1/2}\left(1-\frac{\Delta}{b}\right), \qquad S_1 = \pi\{b^2-(b-\Delta)^2\}, \qquad S_2 = \pi(b-\Delta)^2 \tag{26}$$

Based on elementary mechanics, the energy release rate becomes

$$G^* = \lim_{d\to 0}\left(-\frac{\partial \Pi}{\partial d}\right) = \frac{N^2}{2(E_1S_1+E_2S_2)}\left[\tilde{\gamma}\left(\frac{a}{b}\right)-1\right] \tag{27}$$

where

$$\tilde{\gamma}\left(\frac{a}{b}\right) = \int_0^1 \frac{\mathrm{d}t}{\hat{\gamma}(a/b,t)} = \frac{\lambda/(a/b)}{\sqrt{1-(a/b)^2/\lambda^2}}\arctan\frac{(a/b)}{\sqrt{\lambda^2-(a/b)^2}} \tag{28}$$

From Eqs. (24) and (27), it is not difficult to get

$$K_\mathrm{I} = \sigma_n\sqrt{\pi(b-a)}F\left(\frac{a}{b}\right) \tag{29}$$

where $\sigma_n = N/\pi(b^2-a^2)$ and the normalized stress intensity factor is

$$F\left(\frac{a}{b}\right) = \sqrt{\frac{1-a/b}{a/b}}\left(1+\frac{a}{b}\right)\frac{1}{\lambda}\left\{\frac{1}{2(1-\nu_2^2)}\left[\frac{\lambda/(a/b)}{\sqrt{1-(a/b)^2/\lambda^2}}\arctan\frac{(a/b)}{\sqrt{\lambda^2-(a/b)^2}}-1\right]\right\}^{1/2} \tag{30}$$

For the case of homogeneous cylinders, that is, $\Delta = 0$ or $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = \nu$, $\lambda = 1$. Eq. (30) will become Eq. (13).

6. Theoretical discussions

For cracked cylinders in above sections, two forms of crack surface widening energy release rate are derived based on the concept of three-dimensional G^* integral and elementary mechanics. The former gives a function of $K_{\rm I}$ and the later yields a function of loads. At last, a relationship between stress intensity factors and loads is given.

Actually, the $K_{\rm I}$ for cracked cylinders can also be derived directly from three-dimensional conservation law. Following discussions give the details as an alternative method.

Take the penny-shaped cracked cylinder under tension as an example. Consider a closed curved surface $S_{\rm c} = A + A^+ + A^{\rm cy} + A^-$ for the cracked cylinder shown in Fig. 3. The A^- denotes the remote uncracked cross section; $A^{\rm cy}$ the cylindrical surface and A^+ the cross-section of crack ligament. Along these surfaces, integral values are, respectively,

$$\iint_{A^{\rm cy}} (wn_2 - T_i u_{i,2})\,{\rm d}A = 0 \tag{31}$$

$$\iint_{A^-} (wn_2 - T_i u_{i,2})\,{\rm d}A = \overline{w}^- - N\tilde{u}^-_{2,2} \tag{32}$$

$$\iint_{A^+} (wn_2 - T_i u_{i,2})\,{\rm d}A = -(\overline{w}^+ - N\tilde{u}^+_{2,2}) \tag{33}$$

$$\iint_{A} (wn_2 - T_i u_{i,2})\,{\rm d}A = -\iint_{A_1} w\,{\rm d}A - \frac{a(1-v^2)K_{\rm I}^2}{E} \tag{34}$$

where the $\tilde{u}_i$ shows the displacement of neutral axis. This displacement can be determined by bar theory in mechanics of materials. $\overline{w}$ is the strain energy density per unit length of cylinder. Note that n in above Eqs. (31)–(34) is outward and normal to the surfaces. The outward normal n of A is opposite to x_2 in Eq. (34).

For the closed surface $S_{\rm c} = A + A^+ + A^{\rm cy} + A^-$, the following expression can be given from conservation law

$$\oiint_{S_{\rm c}} (wn_2 - T_i u_{i,2})\,{\rm d}A = \oiint_{A+A^+ + A^{\rm cy}+A^-} (wn_2 - T_i u_{i,2})\,{\rm d}A = 0 \tag{35}$$

Substituting Eqs. (31)–(34) into Eq. (35), the result is

$$\iint_{A_1} w\,{\rm d}A + \frac{a(1-v^2)K_{\rm I}^2}{E} = (\overline{w}^- - N\tilde{u}^-_{2,2}) - (\overline{w}^+ - N\tilde{u}^+_{2,2}) = \frac{N}{2}(\tilde{u}^+_{2,2} - \tilde{u}^-_{2,2}) \tag{36}$$

At remote uncracked cross-section, it is not difficult to get the axial strain from bar theory

$$\tilde{u}^-_{2,2} = \frac{N}{ES} \tag{37}$$

At the cracked cross-section, strain $\tilde{u}^+_{2,2}$ can be found (Xie et al., 1998a,b) from the limit of average strain of oblate spheroid model for penny-shaped cracked cylinder shown in Fig. 5.

$$\tilde{u}^+_{2,2} = \lim_{d\to 0} \frac{1}{d}\int_0^d \frac{N\,{\rm d}x}{ES\left[1-(a/b)^2(1-x^2/d^2)\right]} = \frac{N}{ES(a/b)\sqrt{1-(a/b)^2}} \arctan\frac{a/b}{\sqrt{1-(a/b)^2}} \tag{38}$$

Substituting Eqs. (37) and (38) into Eq. (36), it will yield Eqs. (11) and (12).

Obviously, above discussions can also be applied to the penny-shaped cracked cylinders under lateral loads.

7. Conclusions

In present paper, the relationships between three-dimensional G^* integral and stress intensity factors for center cracked cylinders are derived. An exact method is successfully established by G^* integral which, unlike the J integral, can be applied in an extremely simple manner to some three-dimensional cracks in homogeneous and composite cylinders.

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